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A Forbidden Configuration Theorem of Alon

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DEDICATED TO THE MEMORY OF HERBERT J. RYSER

The paper studies the maximum possible number of distinct rows in a matrix with n columns with entries in column i in $\{0, 1, ..., q_i - 1\}$ that does not contain certain forbidden submatrices. The results might have algorithmic significance if, for example, these matrices are the constraint matrix of a linear program. Combinatorial problems often yield a forbidden submatrix structure. Let $(n; q_1, q_2, ..., q_n)$-matrices be matrices on n columns with entries in column i in $\{0, 1, ..., q_i - 1\}$ and let $\mathscr{S}$ be a family of subsets of $\{1, 2, ..., n\}$. Let $f(n, \mathscr{S})$ be the number of $(n; q_1, q_2, ..., q_n)$-rows which for each $S \in \mathscr{S}$ do not have 0's in all columns S. Noga Alon proved that if A is an $m \times n$ $(n; q_1, q_2, ..., q_n)$-matrix with no repeated rows, and for each $S \in \mathscr{S}$, not all possible rows on columns S, then $m \leqslant f(n, \mathscr{S})$. This paper provides an inductive proof and new $f(n, \mathscr{S}) \times n$ matrices A as above. A linear algebra proof is given for the case $q_1 = q_2 = \cdots = q_n = 2$. Alon's shift proof technique is extended to handle the case A does not have all possible rows on S, each row occurring at least t times. Some other results concerning the extremal $f(n, \mathscr{S}) \times n$ matrices are presented.

1. Introduction

This paper is dedicated to Herb Ryser to whom the author is indebted for much inspiration. Ryser proved the following configuration theorem in 1972. Define a matrix to be *simple* if it has no repeated rows. Define a matrix A to have B as a *configuration* if there is a submatrix of A which is a row and column permutation of B. Finally, define K_k^s to be a $\binom{k}{s} \times k$ (0, 1)-matrix of all rows of s 1's.

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THEOREM 1.1 (Ryser [16]). *Let A be an $m \times n$ simple* (0, 1)*-matrix. Assume A has no configuration*:

$$K_3^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{1.1}$$

Then $m \leqslant \binom{n}{2} + \binom{n}{1} + \binom{n}{0}$ and there exists an A for which equality holds.

In the same year a substantial generalization was proven. Define K_k to be a $2^k \times k$ (0, 1)-matrix of all possible rows on k columns.

THEOREM 1.2 (Sauer [18], Perles and Shelah [19]. *Let A be an $m \times n$ simple* (0, 1)*-matrix. Assume A has no configuration K_k. Then*

$$m \leqslant \binom{n}{k-1} + \binom{n}{k-2} + \cdots + \binom{n}{0}, \tag{1.2}$$

and there exists an A for which equality holds.

Ryser's proof of Theorem 1.1 used linear algebra and the same ideas were used by Frankl and Pach [10] and independently, but later, by the author [4] to provide a linear algebra interpretation of Theorem 1.2. Sauer's proof of Theorem 1.2 is a straightforward induction.

We will use the term *extremal* matrix for one which satisfies a bound as in Theorem 1.2 (1.2) with equality and denote such a bound *best possible* when an extremal matrix exists. The extremal matrix given for Theorem 1.2 was the matrix on n columns with all rows of $k-1$ or fewer 1's. We note that the bound of Theorem 1.1 is a corollary of Theorem 1.2 since K_3^2 is a configuration in K_3. (This idea was exploited by the author [4]). But now Ryser's construction of an extremal matrix is crucial. It was the precursor to the construction of extremal matrices for Theorem 1.2 having no configuration K_k^s (for some chosen s, $0 \leqslant s \leqslant k$) obtained by Füredi and Quinn [12].

A substantial generalization of Theorem 1.2 is due to Alon and provides the focus for this paper. Define an $(n; q_1, q_2, ..., q_n)$-matrix to be a matrix on n columns whose entries in column i belong to $\{0, 1, ..., q_i - 1\}$. To avoid trivialities assume $q_i \geqslant 2$ for all i. Let $\mathcal{S}$ denote a family of subsets of $\{1, 2, ..., n\}$. For an $S \in \mathcal{S}$ with $S = \{s_1, s_2, ..., s_k\}$, $s_1 < s_2 < \cdots < s_k$, define K_S to be a matrix on k columns with all possible $(k; q_{s_1}, q_{s_2}, ..., q_{s_k})$-rows. Note how this differs from $K_{|S|}$ which is a (0, 1)-matrix.

THEOREM 1.3 (Alon [1]). *Let A be an $m \times n$ simple $(n; q_1, q_2, ..., q_n)$-matrix. Assume that for each $S \in \mathscr{S}$, A does not have a row permutation of K_S as a submatrix on columns S. Then*

$$m \leqslant f(n, \mathscr{S}),$$

where $f(n, \mathscr{S})$ is the number of $(n; q_1, q_2, ..., q_n)$-rows which for each $S \in \mathscr{S}$ do not have 0's in all columns S.

Let $M(n, \mathscr{S})$ denote the $f(n, \mathscr{S}) \times n$ simple matrix given by the $f(n, \mathscr{S})$ rows described. Then $M(n, \mathscr{S})$ is an extremal matrix for Theorem 1.3. The result was originally expressed succintly in terms of functions. A result intermediate between Theorems 1.2 and 1.3 was proven by Karpovsky and Milman [14]. Through ignorance and lack of care, a generalization of Theorem 1.2 to $(0, 1, 2, ..., q-1)$-matrices was proven by Murty and the author [7] without proper references being given to Karpovsky and Milman or indeed Alon. The author wishes to apologize. Fortunately, that paper [7] has the redeeming feature of a slick inductive proof and an inductive construction of extremal matrices with no K_k^s configuration.

This paper considers three approaches to Theorem 1.3 of Alon. Section 2 provides an inductive proof. In certain cases extremal matrices exist avoiding $K_{|S|}^t$ in columns $S \in \mathscr{S}$ for a given t where $t \leqslant \min\{|S| : S \in \mathscr{S}\}$. A useful notation for a matrix A and a subset of columns S is $A|_S$ for the submatrix consisting of columns S of A.

Section 3 gives a linear algebra proof of Theorem 1.3 in the case $q_1 = q_2 = \cdots = q_n = 2$, in other words $(0, 1)$-matrices. It is a small generalization of the linear algebra proofs of Theorem 1.2 [10, 4].

Section 4 examines Alon's original proof which shows that an extremal matrix of Theorem 1.3 can be transformed by a shift operator to a monotone matrix. A matrix M is *monotone* if when α, β are two $(n; q_1, q_2, ..., q_n)$-rows with $\alpha \leqslant \beta$ then if α is a row of M, then so is β. A modest generalization is presented to handle the case that the forbidden submatrix on columns S is a row permutation of t copies of K_S. Other consequences of the shift operator are mentioned. Please note that this shift operator was also used by Frankl [9] who has popularized the use of shift operators in extremal set theory.

The paper concludes in Section 5 with discussion of some problems and applications.

2. Inductive Proof and Extremal Matrices

We will provide a proof of Theorem 1.3 using induction on n in the spirit of the proof in [7]. For a family $\mathscr{S}$ of subsets of $\{1, 2, ..., n\}$, define

$$\begin{aligned} \mathscr{S}' &= \{S-n: S \in \mathscr{S}\}, \\ \mathscr{S}'' &= \{S: S \in \mathscr{S}, n \notin S\}. \end{aligned} \tag{2.1}$$

Recalling that $f(n, \mathscr{S})$ denotes the number of $(n; q_1, q_2, ..., q_n)$-rows which for each $S \in \mathscr{S}$ do not have 0's in the columns indexed by S. We may verify the following recurrence

$$f(n, \mathscr{S}) = f(n-1, \mathscr{S}') + (q_n - 1)\, f(n-1, \mathscr{S}''). \tag{2.2}$$

One of the $f(n, \mathscr{S})$ rows which have a nonzero entry in column n (of which there are $q_n - 1$ choices for that entry) cannot have 0's in the columns indexed by S for $S \in \mathscr{S}''$ and so there are $(q_n - 1)\, f(n-1, \mathscr{S}'')$ such rows. Similarly, one of the $f(n, \mathscr{S})$ rows with a 0 in column n cannot have 0's in the columns indexed by S for $S \in \mathscr{S}'$ and so there are $f(n-1, \mathscr{S}')$ such rows.

Proof of Theorem 1.3. The proof is by induction on n. Decompose A as follows by splitting off column n and permuting rows

$$A = \left[\begin{array}{cc} & 0 \\ B_0 & 0 \\ & \vdots \\ & 0 \\ \hline & 1 \\ B_1 & 1 \\ & \vdots \\ & 1 \\ \hline \vdots & \\ \hline & q_n - 1 \\ B_{q_n - 1} & q_n - 1 \\ & q_n - 1 \end{array}\right]. \tag{2.3}$$

Each B_i is a simple $(n-1, q_1 q_2, ..., q_{n-1})$-matrix. Let

$$B = \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_{q_n - 1} \end{bmatrix}. \tag{2.4}$$

Then B has at most $f(n-1, \mathscr{S}'')$ distinct rows by induction. Similarly the number of rows of B_0 common to all q_n blocks B_i is at most $f(n-1, \mathscr{S}')$ since otherwise, by induction, there is an $S \in \mathscr{S}$ so that each B_i has K_{S-n} in columns $S-n$. If $n \notin S$, then this is an immediate contradiction. If $n \in S$, then the rows of K_{S-n} are extended in all q_n possible ways in column n, implying that A has K_S on columns S, a contradiction.

Thus B has at most $f(n-1, \mathscr{S}'')$ rows which may occur at most $q_n - 1$ times with the exception of at most $f(n-1, \mathscr{S}')$ rows which may occur q_n times. The recurrence (1.2) for $f(n, \mathscr{S})$ establishes that $m \leqslant f(n, \mathscr{S})$. ∎

The proof shows that an extremal matrix M for a set $\mathscr{S}$ has an inductive buildup. Decompose M as in (2.3) and form B as in (2.4). Then B has precisely $f(n-1, \mathscr{S}'')$ rows occurring $q_n - 1$ times and $f(n-1, \mathscr{S}')$ rows occurring q_n times. In particular, M contains, in its first $n-1$ columns, extremal matrices for $\mathscr{S}'$ and for $\mathscr{S}''$, demonstrating an inductive buildup. The above proof was motivated by a result in [7] which generalizes slightly.

THEOREM 2.1 [7]. *Let $\mathscr{S}$ be the set of k subsets of $\{1, 2, ..., n\}$. There exists an $f(n, \mathscr{S}) \times n$ simple $(n; q_1, q_2, ..., q_n)$-matrix $A(n, \mathscr{S}, s)$ so that for every $S \in \mathscr{S}$, $A(n, \mathscr{S}, s)$ does not have the configuration K_k^s.*

Proof. The result in [7] was for the case $q_1 = q_2 = \cdots = q_n = q$ but the proof readily adapts to this more general framework. ∎

Another case where we are able to get similarly structured extremal matrices is the case that $\mathscr{S}$ consists of k subsets of $\{1, 2, ..., n\}$ of k consecutive integers. This was a motivating case for the author.

THEOREM 2.2. *Let $\mathscr{S}$ be the family of k subsets of $\{1, 2, ..., n\}$, each consisting of k consecutive integers. Then there exists an $f(n, \mathscr{S}) \times n$ simple $(n; q_1, q_2, ..., q_n)$-matrix $M(n, \mathscr{S}, s)$ with no configuration K_k^s contained in k consecutive columns.*

Proof. Form $M(n, \mathscr{S}, s)$ from the extremal matrix $M(n, \mathscr{S})$ by replacing 1's by 0's and replacing 0's by 1's in those columns i for which $i \equiv 1, 2, ..., s$ (modulo k). By construction for any row of $M(n, \mathscr{S})$ does not have 0's in k consecutive columns. Thus for any k consecutive columns $i, i+1, ..., i+k-1$ any row of $M(n, \mathscr{S}, s)$ does not have a certain $1 \times k$ (0, 1)-row α (with a 1 in column j of α if $i+j-1 \equiv 1, 2, ...,$ or s modulo k) which has precisely s 1's. Thus $M(n, \mathscr{S}, s)$ has no configuration K_k^s contained in k consecutive columns. ∎

3. Linear Algebra Proof

Following the ideas of Ryser's proof of Theorem 1.1, we can obtain a linear algebra proof of Theorem 1.3 in the special case of (0, 1)-matrices, namely, $q_1 = q_2 = \cdots = q_n = 2$. The proof does not seem to extend to the general case. We use the results in [4] rather than the approach of Frankl and Pach [10].

Let $\mathscr{S}$ be a set of subsets of $\{1, 2, ..., n\}$ and let $\mathscr{I}(\mathscr{S})$ denote the subsets of $\{1, 2, ..., n\}$ which for each $S \in \mathscr{S}$ do not contain all of S.

$$\mathscr{I}(\mathscr{S}) = \{T \subseteq \{1, 2, ..., n\}: \text{for all } S \in \mathscr{S},\ T \cap S \neq S\}. \tag{3.1}$$

Note that $|\mathscr{I}(\mathscr{S})| = f(n, \mathscr{S})$ since the (0, 1)-complement of the set-element incidence matrix associated with $\mathscr{I}(\mathscr{S})$ is $M(n, \mathscr{S})$. For any subset $T \subseteq \{1, 2, ..., n\}$ we define the T-fold column intersection vector of an $m \times n$ (0, 1)-matrix A as a vector of length m with a 1 in row i if A has in row i 1's in columns T and we denote this vector as $A(T)$. Note that $A(\emptyset)$ would be the vector of m 1's. For any (0, 1)-vector α, let $n_1(\alpha)$ denote the number of 1's in α.

Theorem 3.1. *Let A be an $m \times n$ (0, 1)-matrix such that for each $S \in \mathscr{S}$, A does not have the configuration $K_{|S|}$ on columns S. Then the number of distinct rows of A is equal to the number of linearly independent (over Q) T-fold column intersection vectors $A(T)$ for $T \in \mathscr{I}(\mathscr{S})$. Thus if A is simple, $m \leqslant f(n, \mathscr{S})$.* ∎

We delay the proof for some preliminary results. An easy induction verifies the following.

Remark 3.2. *Let A, B be (0, 1)-matrices on n columns and assume $n_1(A(T)) = n_1(B(T))$ for all subsets T of $\{1, 2, ..., n\}$. Then A is a row permutation of B.* ∎

For the case that $\mathscr{S}$ is all k-subsets of $\{1, 2, ..., n\}$ we have already proven a result that will be the cornerstone of our proof. Define E_k to be the configuration of all (0, 1)-rows on k columns with an even number of 1's and define O_k to be the configuration of all (0, 1)-rows on k-columns with an odd number of 1's.

Proposition 3.3 [4, Proposition 2.5]. *Let A, B be two (0, 1)-matrices on n columns such that $n_1(A(T)) = n_1(B(T))$ for each subset T of $\{1, 2, ..., n\}$ with $|T| \leqslant k-1$. Assume no row of A is a row of B. Then in some k-subset of columns S, one of A or B has the configuration $E_{|S|}$, and the other matrix contains the configuration $O_{|S|}$.* ∎

We generalize this to arbitrary $\mathscr{S}$.

PROPOSITION 3.4. *Let $\mathscr{S}$ be given and let A, B be two* (0, 1)*-matrices on n columns such that $n_1(A(T)) = n_1(B(T))$ for each $T \in \mathscr{I}(\mathscr{S})$. Assume no row of A is a row of B. Then for some $S \in \mathscr{S}$, in the columns S one of A or B has the configuration $E_{|S|}$ and the other matrix contains the configuration $O_{|S|}$.*

Proof. We say that A, B differ on a set of columns T if the submatrices $A\,|_T$, $B\,|_T$ do not differ only by a row permutation. Choose a minimal T for which this holds. Thus for every subset $U \subseteq T$, $U \neq T$, we have $n_1(A(U)) = n_1(B(U))$ since the submatrices of A, B given by the columns U differ by a row permutation. If there does not exist an $S \in \mathscr{S}$ with $S \subseteq T$ then we deduce that $T \in \mathscr{I}(\mathscr{S})$ and so $n_1(A(T)) = n_1(B(T))$. But then by Remark 3.2, the submatrices $A\,|_T$, $B\,|_T$ are the same, apart from a row permutation, contradicting our choice of T.

Thus we may assume there does exist an $S \in \mathscr{S}$ with $S \subseteq T$. Apply Proposition 3.3 to the submatrices of A, B given by the columns T and with $k = |T|$. We deduce that one of A or B has the configuration $E_{|T|}$ and the other matrix has the configuration $O_{|T|}$. Restricting attention to columns S, we obtain the conclusions of the theorem since the configuration $E_{|T|}$ will contain $E_{|S|}$ in columns S and the configuration $O_{|T|}$ will contain $O_{|S|}$ in columns S. ∎

Proof of Theorem 3.1. Without loss of generality we may suppose A is simple. The number of linearly independent vectors $A(T)$ for $T \in \mathscr{I}(\mathscr{S})$ is at most m. Assume that the number of linearly independent vectors $A(T)$ for $T \in \mathscr{I}(\mathscr{S})$ is less than m and we will obtain a contradiction. Consider the following system of $f(n, \mathscr{S})$ equations in m variables $x_1, x_2, ..., x_m$ with one equation for each $T \in \mathscr{I}(\mathscr{S})$. Let x be the vector of variables:

$$\forall T \in \mathscr{I}(\mathscr{S}), \qquad x \cdot A(T) = 0. \tag{3.2}$$

We have more variables than the rank of the system and so we have a nontrivial integral solution $x^* = (x_1^*, x_2^*, ..., x_m^*)$. Form matrices A_1, A_2 with A_1 containing x_i^* copies of row i of A if $x_i^* > 0$ and A_2 containing $-x_j^*$ copies of row j of A if $x_j^* < 0$. Then A_1, A_2 satisfy the hypothesis of Proposition 3.4 and so for some $S \in \mathscr{S}$, the matrix

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{3.3}$$

contains $K_{|S|}$ on columns S (obtained from an $E_{|S|}$ and $O_{|S|}$ on columns S). But then so does A since $K_{|S|}$ is simple and we have a contradiction which proves the result. ∎

4. A Shift Operator

Alon used the following shift operator for his proof of Theorem 1.3. Shift operators have been used for other forbidden configuration results [9, 11].

Let A be an $(n; q_1, q_2, ..., q_n)$-matrix. Let $1 \leqslant i \leqslant n$, $j+1 \leqslant q_i - 1$. Then $T_{ij}(A)$ is the matrix obtained from A by replacing entries j in column i by $j+1$ as long as the resulting row is not a row of A. Formally we let $x = (a_1, a_2, ..., a_n)$ and define

$$\bar{T}_{ij}(x) = \begin{cases} (a_1, a_2, ..., a_i, ..., a_n) & \text{if} \quad a_i \neq j \\ (a_1, a_2, ..., a_i + 1, ..., a_n) & \text{if} \quad a_i = j. \end{cases} \tag{4.1}$$

Then for A consisting of rows $\alpha_1, \alpha_2, ..., \alpha_m$ we have

$$\text{row } k \text{ of } T_{ij}(A) = \begin{cases} \bar{T}_{ij}(\alpha_k) & \text{if} \quad \bar{T}_{ij}(\alpha_k) \notin \{\alpha_1, \alpha_2, ..., \alpha_m\} \\ \alpha_k & \text{otherwise.} \end{cases} \tag{4.2}$$

Part of Alon's proof is the following result.

PROPOSITION 4.1. *Let i, j satisfy $1 \leqslant i \leqslant n$, $j+1 \leqslant q-1$. Let A be a simple $(n; q_1, q_2, ..., q_n)$-matrix. Then for each $I \subseteq \{1, 2, ..., n\}$, the number of distinct rows in $A|_I$ is greater than or equal that in $T_{ij}(A)|_I$.* ∎

Before going on to a modest generalization of Theorem 1.3, we will consider some simple properties of extremal matrices. Note that we obtain additional extremal matrices from a given extremal matrix by applying a permutation to the entries $0, 1, 2, ..., q_i - 1$ in column i for each i. One special case yields a type of duality. For an $(n; q_1, q_2, ..., q_n)$-matrix $A = (a_{ij})$ we denote the dual of A by $\bar{A} = (\bar{a}_{ij})$, where

$$\bar{a}_{ij} = q_j - a_{ij} - 1. \tag{4.3}$$

Note that $M(n, \mathscr{S})$ is monotone.

THEOREM 4.2. *Let A be a simple $f(n, \mathscr{S}) \times n$ $(n; q_1, q_2, ..., q_n)$-matrix such that for each $S \in \mathscr{S}$, A has no row permutation of K_S on columns S. Then there exist permutation matrices P, Q of order $f(n, \mathscr{S})$ so that*

$$P \cdot \overline{M(n, \mathscr{S})} \leqslant A \leqslant Q \cdot M(n, \mathscr{S}). \tag{4.4}$$

Proof. We follow Alon's proof of Theorem 1.1. Alon notes that by Proposition 4.1, $T_{ij}(A)$ is a simple $f(n, \mathscr{S}) \times n$ $(n; q_1, q_2, ..., q_n)$-matrix and there does not exist an $S \in \mathscr{S}$ with $T_{ij}(A)$ having a row permutation of K_S on columns S. Thus if A is extremal then so is $T_{ij}(A)$. For $A = (a_{ij})$, the sum,

$$\sum_{i=1}^{f(n, \mathscr{S})} \sum_{j=1}^{n} a_{ij} \tag{4.5}$$

strictly increases when we replace A by $T_{ij}(A)$ for $T_{ij}(A) \neq A$. Moreover, if $T_{ij}(A) = A$ for all eligible (i, j) then A is monotone. Thus by a finite sequence of shifts we may obtain from A a monotone extremal matrix which is forced to be a row permutation of $M(n, \mathscr{S})$.

Noting that $A \leqslant T_{ij}(A)$ yields that $A \leqslant Q \cdot M(n, \mathscr{S})$. Duality yields the rest of (4.4). ∎

This result demonstrates the large number of extremal matrices as well as showing the weak structure they contain. Note that, in general, we cannot obtain A from $M(n, \mathscr{S})$ by shifts.

The inequality $P \cdot \overline{M(n, \mathscr{S})} \leqslant A$ has been established in special cases [2, Corollary 4.3; 3, Theorem 6.1] using both the inductive approach of Section 2 and the linear algebra approach of Section 3. Note that Ryser's extremal matrix for Theorem 1.1 was presented in such a way as to suggest the inequality.

We can show the shift operator preserves more than just the forbidden configuration K_S. Define an $(f; p_1, p_2, ..., p_f)$-matrix C to be *multiply reverse monotone* if when α is a row of C occurring t times then for any $(f; p_1, p_2, ..., p_f)$-row β with $\beta \leqslant \alpha$, we have β occurring at least t times in C.

Proposition 4.3. *Let A be an $(n; q_1, q_2, ..., q_n)$-matrix and let C be multiply reverse monotone. Then if $T_{ij}(A)$ has a row permutation of C in columns S then so does A.*

Proof. Assume $T_{ij}(A)$ has a row permutation of C in columns S. Assume C has row α occurring t times. If there is no row β of A with $\bar{T}_{ij}(\beta)\,|_S = \alpha$, where $\beta\,|_S \neq \alpha$, then we deduce that A has t rows containing α in columns S. Otherwise let β be a row of A with $\bar{T}_{ij}(\beta)\,|_S = \alpha$. By the properties of C, the row $\beta\,|_S$ occurs t times in C and so $T_{ij}(A)$ has t rows with $\beta\,|_S$ in columns S. By the definition of T_{ij}, we deduce that A has t rows with $\beta\,|_S$ in columns S which were not altered by the shift operator T_{ij} and so A has t rows with α in columns S. Repeating for all rows α of C finishes the proof. ∎

This result suggests that we may be able to exploit the shift operator further. The following modest generalization of Theorem 1.1 is a result in this direction. Let $f(n, \mathscr{S}, t)$ denote the maximum number of $(n; q_1, q_2, ..., q_n)$-rows so that for each $S \in \mathscr{S}$, there are at most $t-1$ of the rows with 0's in columns S.

Theorem 4.4. *Let A be an $m \times n$ simple $(n; q_1, q_2, ..., q_n)$-matrix. Assume that for each $S \in \mathscr{S}$, A does not have a row permutation of t copies of K_S on columns S. Then*

$$m \leqslant f(n, \mathscr{S}, t). \tag{4.6}$$

Proof. We note that the matrix consisting of t copies of each row of K_S is multiply reverse monotone. Thus we may use Proposition 4.3 to deduce that if A satisfies the hypothesis of the theorem, then $T_{ij}(A)$ is an $m \times n$ simple $(n; q_1, q_2, ..., q_n)$-matrix such that for each $S \in \mathcal{S}$, $T_{ij}(A)$ does not have a row permutation of t copies of K_S on columns S.

Now apply Alon's proof as in Theorem 4.2 to deduce that we may shift A into a monotone $m \times n$ simple $(n; q_1, q_2, ..., q_n)$-matrix M such that for each $S \in \mathcal{S}$, M does not have a row permutation of t copies of K_S on columns S. We deduce inequality (4.6). ∎

In the case $\mathcal{S}$ consists of all k-subsets of $\{1, 2, ..., n\}$, we have for $t > 1$,

$$f(n, \mathcal{S}, t) = \binom{n}{k} + \binom{n}{k-1} + \cdots + \binom{n}{0} + \frac{t-2}{k+1}\binom{n}{k}(1 - o(1)), \quad (4.7)$$

[6, Theorem 3.3], and the monotone extremal matrix for $t = 2$ is unique which is also a result Gronau [13].

5. Problems and Applications

One problem is to devise conditions on the structure of $\mathcal{S}$ so the bound $f(n, \mathcal{S})$ is polynomial in n. Of course, to make sense of this, $\mathcal{S}$ must have some well-defined structure depending on n. In the case $\mathcal{S}$ consists of all k subsets of $\{1, 2, ..., n\}$, Theorem 1.2 shows that $f(n, \mathcal{S})$ is polynomial in n. In the case $\mathcal{S}_n$ consists of subsets of $\{1, 2, ..., n\}$ of k consecutive integers, the bound is exponential in n (for fixed $k > 1$). To be precise $f(n, \mathcal{S}_n)$ satisfies the recurrence relation:

$$\begin{aligned} &f(n, \mathcal{S}_n) = f(n-1, \mathcal{S}_{n-1}) + f(n-2, \mathcal{S}_{n-2}) + \cdots + f(n-k, \mathcal{S}_{n-k}) \\ &f(i, \mathcal{S}_i) = 2^i; \qquad i = 0, 1, ..., k-1. \end{aligned} \quad (5.1)$$

This case was the subject of Theorem 2.2, and the validity of (5.1) is easy to establish by splitting up a row on n columns, with no k consecutive 0's, at the rightmost 1.

The problem remains of extending the linear algebra proof of Theorem 3.1. to the general case. One potential way to apply the ideas is in the following sort of result obtained from Proposition 3.4.

THEOREM 5.1. *Let $\mathcal{S}$ be a set of subsets of $\{1, 2, ..., n\}$ and let A, B be $(0, 1)$-matrices on n columns such that $n_1(A(T)) = n_1(B(T))$ for $T \in \mathcal{I}(\mathcal{S})$. Assume for each $S \in \mathcal{S}$, $A\mid_S$ has no configuration $E_{|S|}$ or $O_{|S|}$. Then B is a row permutation of A.*

Note how the result provides a potentially compact way of encoding A by the entries $n_1(A(T))$ for $T \in \mathscr{I}(\mathscr{S})$. This generalizes results of Ryser [15, Theorem 4.1; 17, Theorem 3.1], and was discussed as an encoding scheme in a special case in [5]. Of course, Proposition 3.4 could be used directly. Theorem 5.1 does raise the question of how many distinct rows A can have.

Let $t(n, k, k-1)$ be the Turan number denoting the maximum number of rows of $k-1$ 1's in a simple (0, 1)-matrix A on n columns so that A has no configuration K_k^{k-1}. The exact value for $t(n, k, k-1)$ is only known for $k=3$ but estimates are known [8]. The following results expands on Remark 3 of Alon [1] and gives it a forbidden configuration application. See also Frankl [9].

THEOREM 5.2. *Let A be an $m \times n$ simple* (0, 1)*-matrix with no configuration E_k or O_k. Then the best possible bound on m is*

$$t(n, k, k-1) + \binom{n}{k-2} + \binom{n}{k-3} + \cdots + \binom{n}{0}. \tag{5.2}$$

Proof. Let B be a monotone matrix obtained from A by shifts in the manner of the proof of Theorem 4.2 (or see Alon [1]). The forbidden configurations E_k, O_k ensure that for each k-subset S of columns that $A|_S$ has at most $2^k - 2$ distinct rows. By Proposition 4.1, B has this property and since B is monotone $B|_S$ does not have the row of 0's and is missing at least one possible row of one 1. Thus B has no configurations K_k^0, K_k^1 or indeed E_k, O_k.

We deduce that B has row sums at least $n-k+1$ and that those rows of row sum $n-k+1$ form a matrix with no configuration K_k^1. By taking (0, 1)-complements, we deduce that B has at most $t(n, k, k-1)$ rows of row sum $n-k+1$. Trivially, B has at most $\binom{n}{k-2} + \binom{n}{k-3} + \cdots + \binom{n}{0}$ rows of row sum more than $n-k+1$. Thus m is at most (5.2).

We see that (5.2) is best possible by constructing B from all rows of more than $n-k+1$ 1's and the (0, 1)-complements of the $t(n, k, k-1)$ rows of $k-1$ 1's avoiding the configuration K_k^{k-1}. ∎

Further work should provide additional applications of Theorem 1.3, particularly using the full generality of $\mathscr{S}$. A potential application of the generality of $\mathscr{S}$ is to 3-dimensional matrices, for example:

PROPOSITION 5.3. *Let A be an $m \times n \times p$* (0, 1)*-array with no repeated $n \times p$ planes. Assume A has no $2^{kl} \times k \times l$ subarray of all possible $k \times l$(0, 1)-planes. Then $M \leqslant f(np, \mathscr{S})$, where $\mathscr{S}$ is the set of subsets $\{i_r + j_s p$: $r = 1, 2, ..., k$; $s = 1, 2, ..., l\}$ taken over all k subsets $\{i_1, i_2, ..., i_k\}$ of $\{1, 2, ..., n\}$ and all l subsets $\{j_1, j_2, ..., j_l\}$ of $\{0, 1, ..., p-1\}$.*

The following amusing variation of Theorem 1.2 follows from Theorem 1.3. Perhaps such results will have applications in further configuration investigations.

PROPOSITION 5.4. *Let A be an $m \times n$ simple $(0, 1)$-matrix with $m > \binom{n}{k-1} + \binom{n}{k-2} + \cdots + \binom{n}{0} + 1$. There are two distinct k-subsets of columns S, T so that $A|_S$ and $A|_T$ both contain the configuration K_k.*

Proof. Use Theorem 1.3 with $\mathscr{S}$ consisting of all but one k-subset of $\{1, 2, ..., n\}$. ∎

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